In conclusion we mention the unusual course of deformation of the region of flow with increasing φ_+ in the case when a reverse current exists even when $\varphi_+ = 0$ (e.g. when $a = 0.4$, $\beta = 4$). When φ_+ increases, the region of flow varies from one infinitely large as the singularity first appears at infinity, to one localized in the central part at $y = 0$.

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WAVES OF IONIZATION AND RECOMBINATION IN A WEAKLY IONIZED MAGNETIZED PLASMA

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It is established that several types of the ionization and recombination waves may occur in a weakly ionized plasma containing additions of an easily ionizable component, in the presence of a magnetic field. The rate of propagation of such waves are determined by the conditions arising from the fact that the waves have certain structures. It is shown that the waves considered in the present paper satisfy the criterion of evolutionarity.

The problem of propagation of an ionization wave through a weakly ionized plasma in the absence of a magnetic field was first posed in [1]. The expression obtained there for the rate of propagation of the ionization wave was confirmed experimentally in [2], but at the same time the experimental data obtained in [3, 4] disagreed with the results of [1]. The analysis in [1] was performed for a model of two-fluid hydrodynamics and it was assumed that the plasma is in the state of ionization equilibrium. A motion of the ionization wave during which this equilibrium is disturbed, was studied in [5]. A study of the structure of the ionization shock waves (the conductivity in such waves varies from zero to infinity) in a single-fluid model of magnetohydrodynamics was initiated in [6] and taken up by a number of authors (see survey [7]). A comprehensive survey of the ionization discontinuities of various types is also given in [8].

Below we study the ionization and recombination waves in a plasma containing additions of an easily ionizable component when the influence of the Hall effect is significant and the temperature of electrons exceeds the temperature of the heavy particles by many times.

1. Let us consider a plasma composed of heavy particles (atoms and ions) and electrons, assuming that the parameters of the heavy particles (pressure, temperature T and density) are known and vary little within the characteristic domain of strong variation of the basic electron parameters. We assume that the time in which the ionization equilibrium is attained, is considerably shorter than the characteristic time of the problem, and for this reason the electron concentration n and their temperature T_e depend on each other

$$n = n(T_e) \tag{1.1}$$

In the simplest case this relation can be obtained using the Saha equation.

A system of equations describing the state of the medium has the form

$$\begin{gathered}
\operatorname{div} \mathbf{j} = 0, \qquad \operatorname{rot} \mathbf{E} = 0, \qquad \mathbf{j} + [\mathbf{j}\Omega] = \sigma \mathbf{E} \\
I\left(\frac{\partial n}{\partial t} + \mathbf{u}_e \nabla n\right) + \operatorname{div} \mathbf{q} = \frac{j^2}{\sigma} - N_- \\
N_- = {}^3/_2 k \delta T_e n \frac{1}{\tau}, \quad \mathbf{q} + [\mathbf{q}\Omega] = -\lambda \nabla T_e \\
\tau = \left(\sum_r n_r Q_{er} V\right)^{-1}, \quad \Omega = \frac{e}{m_e} B\tau
\end{gathered} \tag{1.2}$$

Here j is the electric current density, E is the electric field strength, τ is the time of collision of an electron with the remaining plasma particles, Q_{er} is the collision cross section between the electron and the particles of the type r, Ω is the Hall parameter, B is the magnetic field induction, the coefficients of the electrical $\sigma(n, T_e)$ and the heat $\lambda(n, T_e)$ conductivity are known functions of the electron temperature and concentration, U_e is the velocity of electrons, δ is the amount of energy given up by the electrons colliding with the heavy particles (in general δ is a function of the temperature of both, the electrons and the heavy particles). When the medium in question contains an addition of an easily ionizable component and the degree of ionization of the basic component is small in the range of temperatures considered, I is the ionization potential of the additive. The effect of the induced field is neglected and the constant external magnetic field is directed along the z-axis.

The manner in which the radiation effects exert their influence, depends on the composition of the plasma and on the temperature range. If the losses due to radiation can be described in an algebraic form, then this simply alters the form of the function N_- in the fourth equation of (1.2). If on the other hand the radiation is accounted for by means of the radiant thermal conductivity, then the functional dependence $\lambda = \lambda(n, T_e)$ itself is altered. The method of constructing the solution which is discussed below remains the same, and the influence of radiation is neglected in the present investigation.

The system (1.1), (1.2) admits a homogeneous solution stationary over the whole region and this solution is obtained by equating to zero the right-hand side of the fourth equation of (1.2). If in addition we linearize the system (1.1), (1.2), then for the values of the Hall parameter exceeding its critical value Ω_-, this stationary state is unstable and an ionization-type instability develops in the plasma [9]. We shall bypass the problem of the manner in which the ionization-type instability develops in the presence of nonlinear effects (this was dealt with in [10]) and consider the types of the ionization and recombination waves in a plasma containing an easily ionizable component. The electron concentration (electron temperature) in these waves varies. Obviously, these waves resemble the ionizing shock waves discussed in [6] except for the fact that the conductivity of the medium behind such wave fronts is finite and the heavy particle parameters do not vary.

We shall assume that the surface of the discontinuity coincides with the yoz-plane and that the parameters of the medium vary in the normal direction which in the present case coincides with the x-axis. In this case the first and second equation of (1.2) express the law of conservation of the normal component of the electric current density and of the tangential electric field component during the passage across the surface of discontinuity

$$j_x \equiv j_0 = \text{const} < 0, \; E \equiv E_0 = \text{const} \tag{1.3}$$

From the third and fourth equation of (1.2) we find, with (1.3) taken into account, that the structure of such waves is described by

$$I\left(\frac{\partial n}{\partial t} - \frac{j_0}{en}\frac{\partial n}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\lambda}{1+\Omega^2}\right)\frac{\partial T_e}{\partial n}\frac{\partial n}{\partial x} = N_+\frac{j_0^2}{\sigma} - N_- \tag{1.4}$$

$$N_+ = 1 + \Omega^2 + \left(\frac{\sigma E_0}{j_0^2}\right)^2 + 2\sigma\Omega\frac{E_0}{j_0}$$

If the length of the transition zone is small, we can consider an infinitely thin surface of discontinuity at which the parameters j_y, E_x, n and T_e vary in jumps. It is clear that the relations obtained from the third equation of (1.2) and (1.1), $uj_0^2 N_+ / \sigma = N_-$ are insufficient to determine the four parameters of the medium ahead (behind) the wave and the velocity of motion W of the wave itself, by the known parameters of the medium behind (ahead) the wave. Additional relations (in the present case a single relation) are determined by analysing the wave structure [11].

Having obtained n_0 and T_0 from the relation $j_0^2 = \sigma_0 N_-(n_0, T_0)$, we introduce the dimensionless parameters

$$s = \frac{n}{n_0}, \quad \Lambda = \lambda\frac{\partial T_e}{\partial n}\frac{n_0}{\lambda_0 T_0}, \quad L^{-2} = \frac{j_0^2}{\sigma_0 n_0 T_0}, \quad t^+ = \frac{In_0\sigma_0}{j_0^2}, \quad U^+ = \frac{L}{t^+} \tag{1.5}$$

$$U = \frac{j_0}{en_0u^+}, \quad x_- = \frac{x}{L}, \quad t_- = \frac{t}{t^+}$$

and seek the solution of (1.4) in the form

$$s = s(\xi), \; \xi = x_- - Wt_-, \; W > 0, \; j_0 < 0$$

In this case the wave structure is described by the following equation and the boundary conditions:

$$\frac{d}{d\xi}\frac{\Lambda(s)}{1+\Omega^2}\frac{ds}{d\xi} - (U - W)\frac{ds}{d\xi} + F = 0 \tag{1.6}$$

$$F = N_+ - N_-\sigma_0 / j_0^2$$

$$\frac{ds}{d\xi} \to 0, \quad x \to \pm\infty \tag{1.7}$$
$$s(\infty) = s_i,\ s(-\infty) = s_j,\ i \neq j$$

Clearly, the states ahead and behind the wave represent the equilibrium points of (1.6), or of equivalent system of ordinary differential equations

$$\frac{ds}{d\xi} = (1 + \Omega^2)\frac{P}{\Lambda} \tag{1.8}$$
$$\frac{dP}{d\xi} = \frac{1 + \Omega^2}{\Lambda}(U - W)P - F$$

2. The equilibrium points of system (1.8) are zeros of the function F. The number of equilibrium points, their indices and the values of the plasma parameters at an equilibrium point all depend on j_0, E, the magnetic field induction (Hall parameter) and the composition of the plasma. The composition of the plasma influences the position of the equilibrium points by virtue of the specific relationship connecting it with the collision cross section of the electrons and heavy particles, and hence through the thermal conductivity and electric conductance. Let us consider, for definiteness, the temperature range from 10^3 to 10^4 °K and the argon plasma containing an addition of easily ionizable cesium. When the magnetic field is absent, two equilibrium points exist, the node $(s_0, 0)$ and the saddle point $(s_1, 0)$. A separatrix emerging from the singularity $(s_0, 0)$ and arriving at $(s_1, 0)$ corresponds to the ionization wave. The electron concentration in the wave varies from zero to some finite value. The structure of such a wave was investigated in [1] and shall not be discussed here.

When the Hall parameter reaches a certain value Ω_-, the solution bifurcates and four equilibrium points s_j, $j = 0,1,2,3$ exist for $\Omega_+ > \Omega > \Omega_-$. When $\Omega < \Omega_-$, the number and the character of the singularities remain unchanged. In the range $T_e > T_+$

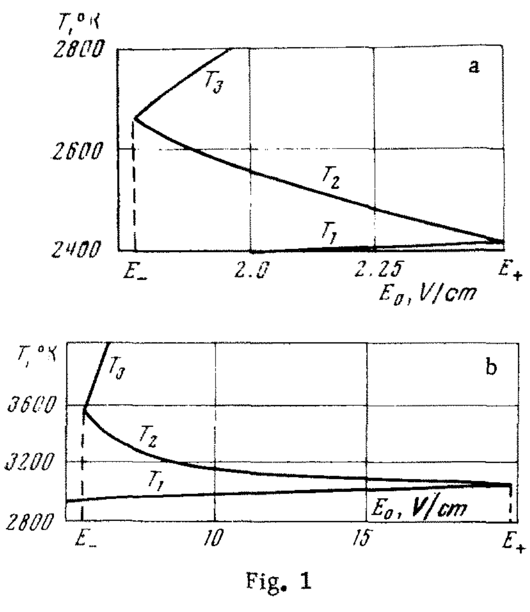

Fig. 1

we have, for $\Omega_- < \Omega < \Omega_+$ three equilibrium points s_1, s_2 and s_3. When $\Omega > \Omega_+$ these points again reduce, in the given temperature range, to a single point which is a saddle point. A bifurcation diagram showing the dependence of the equilibrium points of (1.8) on the electric field $(B_0 = \text{const})$ is given in Fig. 1 which depicts the dependence of the electron temperatures (T_1, T_2, T_3) at the equilibrium points on the magnitude of the electric field for the argon plasma containing cesium, for $B = 0.3$ tesla (a) and $B = 2.0$ tesla (b). When $E < E_-$, a single saddle-type equilibrium point exists, while for $E_- < E < E_+$ there are three equilibrium points. The points $(s_1, 0)$ and $(s_3, 0)$ are saddle points and $(s_2, 0)$ can be either a node or a focus; the focus can be either stable or unstable. The latter property is determined by the rate of propagation of the wave front.

From the study of the character of the phase trajectories it follows that when $j_0 < 0$ and $W > 0$, two types of ionization waves are possible: the first type corresponds to the passage from $(s_1, 0)$ to $(s_3, 0)$, and the second type to the passage from $(s_2, 0)$ to $(s_3, 0)$. If the point $(s_2, 0)$ is a node, then the ionization wave has a monotonous character; if on the other hand $(s_2, 0)$ is an unstable focus, then the wave structure is of oscillatory character. Considering the wave structure we find that the velocity of the second type wave (passage from $(s_2, 0)$ to $(s_3, 0)$) is smaller than the velocity of the first type wave.

The electron velocity U_e is the characteristic velocity of propagation of small perturbations of the system (1.2). Introducing the quantity $m_i = |W / U_{ei}|$ equal to the ratio of the velocity of propagation of the wave front to the velocity of propagation of small perturbations in the corresponding homogeneous state, we find that from the ionization waves of the first type we have $m_1 > 1$ and $m_2 < 1$. The subscript (1) describes the state ahead the wave and (2), the state behind the wave. For the second type waves we have

$$m_1 < m_2 < 1.$$

The waves differ physically from each other in that the first type waves undergo a stronger jump in the value of the electron concentration (the temperature jump) than the second type waves, and their wave front propagates at a greater rate. Which of the possible types of the ionization waves is realized under the given conditions depends on the external parameters given and on the boundary conditions that have to be satisfied. Considering the phase trajectories we find that the passage from the point $(s_1, 0)$ when it is a stable focus to $(s_2, 0)$, is not possible. Moreover, we do not consider the case when the system (1.6) admits solutions different from the ionization or recombination waves, since all these problems as well as the case of $j_0 > 0$ and $W < 0$, were investigated in [10].

Let us consider certain possible types of recombination waves characterized by a drop in the concentration and temperature of the electrons during the passage across the wave surface. Waves of this type correspond to the passages from $(s_3, 0)$ to $(s_1, 0)$ (first type) and from $(s_3, 0)$ to $(s_2, 0)$ (second type). For the recombination waves of the first type we have $m_1 > 1$ and $m_2 < 1$ and for the second type we have $1 < m_2 < m_1$.

3. When the plasma is not in the state of ionization equilibrium, Eq. (1.1) is invalid and must be replaced by another equation describing the kinetics of ionization and recombination

$$\frac{\partial n}{\partial t}+U_e\frac{\partial n}{\partial x}-\frac{\partial}{\partial x}\frac{D_e}{1+\Omega^2}\frac{\partial n}{\partial x}=M_+-M_- \tag{3.1}$$

$$n_{\text{H}}=n_a+n_i$$

Here n_{H} is the initial concentration of the additive, M_+ and M_- denote the intensity of the sources of production and annihilation of electrons. In its general form the equation may become very cumbersome, therefore we shall limit ourselves to the case when the ionization takes place due to the impact of electrons, and the recombination due to triple collisions. We have

$$M_+-M_-=k_r\,(Rn_an_e-n_e^3) \tag{3.2}$$

$$R=A\left(\frac{2\pi m_e}{h^2}\right)^{3/2}T_e^{3/2}\exp\left[-\frac{I}{kT_e}\right]$$

where R is the equilibrium constant of the reaction and k_r is the recombination coefficient which is a known function of the plasma parameters.

The structure of the ionization or recombination waves is described by the following equations:

$$-(W-U)\frac{ds}{d\xi}-\frac{d\Delta_1}{d\xi}\frac{1}{1+\Omega^2}\frac{ds}{d\xi}=k_r^+\,(R^+n^+s-s^3) \tag{3.3}$$

$$-(W-U)\frac{ds}{d\xi}-\frac{d}{d\xi}\frac{\Lambda}{1+\Omega^2}\frac{d\theta}{d\xi}=F$$

$$\Delta_1=\frac{D_et^+}{L^2},\qquad \Lambda=\frac{\lambda}{\lambda_0}$$

$$\theta=\frac{T}{T_0},\quad k^+=\frac{k_rn_0}{t^+},\quad R^+=\frac{R}{n_0}$$

where $k_r^+\sim\tau_x/\tau_u$ is the ratio of the time of the Joule dissipation heating to the time of ionization.

From the above relationships it follows that the equilibrium points of the systems (1.6) and (3.3) coincide, since the Saha equation follows from the condition $M_-=M_+$. Consequently everything that has been said in Sect. 2 about the number of singularities and their character remains valid, the only change being in the character of the distribution of the plasma parameters within the wave structure.

4. We shall show that the ionization and recombination waves of the second type discussed above satisfy the evolutionarity condition. This condition states that the number of relations on the surface of discontinuity must exceed the number of small perturbation waves moving away from the discontinuity, by unity. The equations must suffice for determination of the amplitudes of the departing waves and the wave front velocity in terms of the known amplitudes of the waves incident on the surface of discontinuity [12]. In fact, for both, the ionization wave and recombination wave (of the second type), there exists a single wave of small perturbations, moving away from the surface of discontinuity. For the ionization wave this perturbation wave moves upstream, and for the recombination wave it moves downstream. The number of relationships on the surface of discontinuity which can be written in the form

$$F=0,\ n=n\,(T_e),\ W=W\,(s,\ldots) \tag{4.1}$$

is sufficient for determination of the amplitude of the wave of small perturbations, moving away from the surface of discontinuity (in the given case – the perturbations of the electron concentration and their temperature), and the rate of propagation $W^{(1)}$ of the discontinuity. The third equation of (4.1) connects the rate of motion of the wave front with the parameters on both sides of the discontinuity, and is obtained from the condition of existence of the wave structure.

In addition to the moving waves, the standing ionization or recombination waves may exist. For such waves $W = 0$ and the point $(s_2, 0)$ is a stable node or focus.

A general solution of the problem satisfying Eqs. (1.1) and (1.2) for two-dimensional or three-dimensional regions may be composed of several regions with constant or continuously varying parameters, separated by moving or standing ionization or recombination waves. For example, the case depicted in Fig. 2 (phase curve for $E = 2$ V/cm, $B = 0.3$ tesla, argon plasma with addition of cesium, $\theta = T_e / T_x$, $T_x = 2593$ °K) can correspond to the ionization wave ABC and the recombination wave CDA following each other. The parameters of the medium are constant between these two waves.

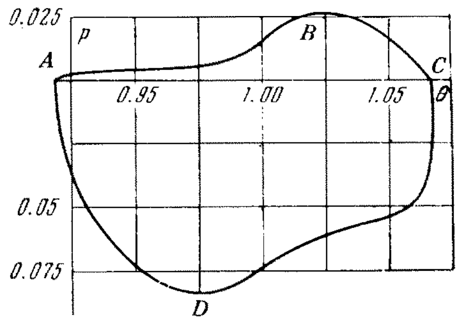

Fig. 2

Investigation of the stability of the ionization and recombination waves of the first and second type has shown that these waves have a stable structure.

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TWO-DIMENSIONAL PROBLEM OF THE MOTION OF A SNOW AVALANCHE ALONG A SLOPE WITH SMOOTHLY CHANGING PROPERTIES

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To describe the motion of an avalanche we use "hydraulic" models, one version of which for a one-dimensional motion was proposed in [1]. An asymptotic solution as $t \to \infty$ was constructed in [2] for the equations proposed in [1] for the case of a slope of constant steepness with a uniform snow cover. Below we investigate the asymptotic behavior of the solution of a two-dimensional problem of the motion of a snow avalanche along a slope of varying steepness, on which snow with varying properties lies. It is assumed that the typical linear scale of variation of these quantities is sufficiently large.

1. Statement of the problem. The equations of two-dimensional motion of a snow avalanche, analogous to those proposed in [1] for the one-dimensional case, are written in the form

$$dh / dt + h \operatorname{div} \mathbf{v} = 0 \tag{1.1}$$

$$\frac{d\mathbf{v}}{dt} = -\frac{g}{2h} \operatorname{grad}(h^2 \cos\psi) + \mathbf{e} g \sin\psi - F(v, h, x, y)\mathbf{v} \tag{1.2}$$

Here v is the snow's velocity averaged over the thickness, h is the thickness of the moving snow layer, ψ is the angle between the horizontal plane and the tangent plane to the slope at a point being considered, $\mathbf{e}$ is a vector lying in the tangent plane and